12/Mar/2019

UNIT- IV

Frequency response Analysis

The ratio of sinusoidal response and sinusoidal input is called sinusoidal transferfunction, and in general it is denoted by $T(j\omega)$. The sinusoidal transferfunction is the frequency domain representation of the system and so it is also called frequency domain Transferfunction.

If the 'S' domain transferfunction $T(s)$ is known the frequency domain transferfunction $T(j\omega)$ can be obtained directly from $T(s)$ by replacing 's' by 'jω' ie $T(s) \Rightarrow T(j\omega)$

$$r(t) \longrightarrow \boxed{|T(j\omega)| \angle (T(j\omega))} \longrightarrow c(t) = B\angle\theta$$

$r(t) = A(\sin\omega t + \theta)$

$= A\angle\theta$

$B\angle\theta = A|T(j\omega)| \angle(\theta + T(j\omega))$

* Frequency Response :-

The frequency domain Transferfunction $T(j\omega)$ is a complex function of 'ω'. Hence it can be seperated into magnitude function and phase function. Now the magnitude and phase functions will be real functions of 'ω' and they are called frequency response. The frequency response for

open loop function: $G(s) \xleftrightarrow{s=j\omega} G(j\omega) \angle G(j\omega)$

loop function: $G(s)H(s) \xleftrightarrow{s=j\omega} |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$

closed loop function: $M(s) \xleftrightarrow{s=j\omega} M(j\omega) \angle M(j\omega)$.

For unity feedback system $H(s) = 1$, and open loop, loop transfer functions are same.

→ Advantages of frequency response analysis:-

1) The absolute and relative stability of closed loop system can be estimated from the knowledge of open loop frequency response.

2) The pratical testing of systems can be easily carried with available sinusoidal signal generators, and

measuring Equipments.

3) The transfer function of complicated systems can determined experimentally by frequency response test.

4) The design and parameter adjustment of open loop transferfunction of a system for a specified closed loop performance is carried out more easily in frequency domain.

5) when the system is designed by the use of frequency response analysis the effect of Noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.

6) The frequency response analysis and designs can be extended to certain non-linear control systems.

→ Disadvantages:-

1) Basically the Methods can be applied only to linear systems. when these methods are applied to the systems of Non-linearity the result of analysis and design are not exact.

2) These methods are consider somewhat old and outdated in view of advanced methods developed for digital computer simulation and Modeling.

3) obtaining frequency response is possible only if the time constants are upto few minites.

4) frequency response pratically time consuming method.

* Frequency domain specifications:-

1) Responant peak (M_r):-

The Maximum value of the Magnitude of closed loop transferfunction is called Resonant peak.

2) Resonant frequency (ω_r):-

The frequency at which resonant peak occurs is called resonant frequency.

3) Band width (ω_b) :-

A Band width is the range of frequencies for which the system normalized gain is more than -3db. The frequency at which the gain is -3db is called cutoff frequency. Band width is usually defined for closed loop system and transmits the signals whose frequencies are less than cutoff frequency.

4) cut-off rate :-

The slope of the log-magnitude curve nearly cutoff frequency is called cutoff rate.

5) Gain Margine :- (K_g) ☆

The gain margine is defined as value of gain to be added to the system in order to bring the system to the verge of instability. The gain margine (K_g) is given by reciprocal of magnitude of open loop transfer function at phase cross over frequency.

The frequency at which the phase of open loop transfer function is 180° is called phase cross over frequency. (ω_{pc}). $\therefore$ gain margine $K_g = \dfrac{1}{|G(j\omega_{pc})|}$

The gain margin in db can be expressed as

$$K_g \text{ in dB} = 20\log K_g = 20\log\left[\frac{1}{|G(j\omega_{pc})|}\right]$$

6) phase Margine :- ☆ (γ)

The phase margine 'γ' is defined as additional phase lack to be added at the gain crossover frequency in order to bring the system to the verge of instability. The gain ~~fre~~ crossover frequency ω_{gc} is the frequency at which the magnitude of open loop transferfunction is unity. phase margine $\gamma = 180 + \phi_{gc}$.

(13/Mar/2019) * **Bode plot** :-

The Bode plot is a frequency response plot of the sinusoidal transferfunction the system. A bode plot consists of two graphs one a plot of magnitude of sinusoidal transferfunction Vs $\log \omega$. The other a plot of phase angle of sinusoidal transfer ction Vs $\log\omega$. The bode plot can be drawn both open loop & closed loop system. Usually bode plot is drawn for open loop system.

Consider open loop transfer function

$$G(S) = K(1+ST_1)/S(1+ST_2)(1+ST_3)$$

$j\omega = \omega$,
$(j\omega)^2 = \omega^2$
$(j\omega)^3 = \omega^3$

$$G(j\omega) = K(1+j\omega T_1)/j\omega(1+j\omega T_2)(1+j\omega T_3)$$

Taking Modulus

$$|G(j\omega)| = \frac{K\sqrt{1+\omega^2 T_1^2} \angle \tan^{-1}\left(\frac{\omega T_1}{1}\right)}{\omega\angle 90^\circ (\sqrt{1+\omega^2 T_2^2}) \angle \tan^{-1}\left(\frac{\omega T_2}{1}\right) (\sqrt{1+\omega^2 T_3^2}) \angle \tan^{-1}}$$

considering only magnitude.

$$|G(j\omega)| = K\sqrt{(1+\omega^2 T_1^2)}/\omega(\sqrt{1+\omega^2 T_2^2})(\sqrt{1+\omega^2 T_3^2})$$

considering only phase

$$\angle G(j\omega) = \tan^{-1}(\omega T_1) - 90^\circ - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$$|G(j\omega)| \text{ in dB} = 20\log|G(j\omega)|$$

$$= 20\log\left[\frac{K\sqrt{1+\omega^2 T_1^2}}{\omega(\sqrt{1+\omega^2 T_2^2})(\sqrt{1+\omega^2 T_3^2})}\right]$$

$$= 20\log\left(\frac{K}{\omega}\right) + 20\log(\sqrt{1+\omega^2 T_1^2}) + 20\log\left[\frac{1}{\sqrt{1+\omega^2 T_2^2}}\right] + 20\log\left[\frac{1}{\sqrt{1+\omega^2 T_3^2}}\right]$$

① Sketch Bode plot for the following transfer function and determine the system gain 'K' for the gain cross over frequency to be 5 rad/s

$$G(S) = KS^2/(1+0.2S)(1+0.02S)$$

Q1 $G(j\omega) = K(j\omega)^2 / (1+j0.2\omega)(1+j0.02\omega)$

Assume $K=1$ in bode plot.

Mag $|G(j\omega)| = K\omega^2 / (\sqrt{1+(0.2\omega)^2})(\sqrt{1+(0.02\omega)^2}$

pha $(\angle G(j\omega)) = 180^\circ - \tan^{-1}\left[\frac{0.2\omega}{1}\right] - \tan^{-1}\left(\frac{0.02\omega}{1}\right)$

magnitud plot :-

let the corner frequency

$\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$ $\omega_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec}.$

The various terms of $G(j\omega)$ are listed in the following table in the increasing order of their corner frequencies also the tables shows the slope contributed by each term and changing slope at the corner frequency.

Terms	corner freq	slope(dB/dec)	change in slope (dB/dec)
$(j\omega)^2$	do not exist	+40	
$\left(\frac{1}{1+j0.2\omega}\right)$	$\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$	−20	20
$\left(\frac{1}{1+j0.02\omega}\right)$	$\omega_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec}$	−20	0

slope of $(j\omega) = 20$, slope of $\left(\frac{1}{j\omega}\right) = -20$,

slope of $(j\omega)^2 = 40$, slope of $\left(\frac{1}{j\omega}\right)^2 = -40$

chose a lower frequency ω_L and a higher frequency ω_H such that $\omega_l < \omega_{c1}$ and $\omega_h > \omega_{c2}$

let $\omega_l = 0.5$ rad/sec and $\omega_h = 100$ rad/sec.

let $A = |G(j\omega)|$ in dB. let us calculate at ω_l, ω_{c1}, ω_{c2}, ω_h

$\omega = \omega_l$; $A = 20\log|(j\omega)^2| = 20\log|(0.5)^2| = -12\text{dB}$

$\omega = \omega_{c1}$; $A = 20\log|(j\omega)^2| = 20\log|(5)^2| = 27.96 \approx 28\text{dB}$

$\omega = \omega_{c2}$; $A = \left[(\text{slope from } \omega_{c1} \text{ to } \omega_{c2})\left(\log\frac{\omega_{c2}}{\omega_{c1}}\right)\right] + A(\omega = \omega_{c1})$

$= 20 \times \log\left(\frac{50}{5}\right) + 28 = 48\text{dB}$

$\omega = \omega_h$; $A = \left[(\text{slope from } \omega_{c2} \text{ to } \omega_h)\left(\log\frac{\omega_h}{\omega_{c2}}\right)\right] + A(\omega = \omega_{c2})$

$= \left[0 \times \log\left(\frac{100}{50}\right)\right] + 48 = 48\text{dB}.$

Phasor plot :-

$\angle G(j\omega) = 180 - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$ degrees

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}(0.02\omega)$	$\angle G(j\omega)$	Points on Graph
0.5	5.7	0.6	174	a
1	11.3	1.1	168	b
5	45	5.7	130	c
10	63.4	11.3	106	d
50	84.3	45	50	e
100	87.1	63.4	30	f

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2) Sketch bode plot and determine phase margine and gain margine, where $G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$

Sol

We know that $s^2 + 2\xi\omega_n + \omega_n^2$

from problem $\omega_n^2 = 100 \Rightarrow \omega_n = 10$, $2\xi\omega_n = 16 \Rightarrow$

$$G(s) = \frac{75(1+0.2s)}{s\times100\left(\frac{s^2}{100} + \frac{16}{100}s + 1\right)} = \frac{0.75(1+0.2s)}{s(0.01s^2 + 0.16s + 1)}$$

Replace s by $j\omega$

$$G(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega(0.01j\omega^2 + 0.16j\omega + 1)} = \frac{0.75(1+0.2j\omega)}{j\omega[(1-0.01\omega^2) + 0.16j\omega]}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0.2\omega}{1}\right) - 90^\circ - \tan^{-1}\left[\frac{0.16\omega}{(1-0.01\omega^2)}\right]$$

Magnitude plot:-

let the corner frequencies

$\omega_{c_1} = \frac{1}{0.2} = 5$ rad/sec $\qquad \omega_{c_2} = 10$ rad/sec.

Note: for second order system $\omega_{c_2} = \omega_n = 10$

Terms	corner freq	slope dB/dec	change in the slope
$\frac{0.75}{j\omega}$		-20	
$(1+0.2j\omega)$	$\omega_{c_1} = 5$ rad/sec	$+20$	$-20+20 = 0$
$\frac{1}{(1-0.01\omega^2)+j0.16}$	$\omega_{c_2} = 10$ rad/sec	-40	$0-40 = -40$

choose a lower frequency such that $\omega_l < \omega_{c1}$ and a higher frequency ω_h such that $\omega_h > \omega_{c2}$

ie $\omega_l = 0.5$, $\omega_h = 100$

at $\omega = \omega_l$; $A = 20\log|G(j\omega)| = 20\log\left[\frac{0.75}{j\omega}\right]_{j\omega = 0.5}$

$$A = 20\log\left[\frac{0.75}{0.5}\right] = +3.52 \text{ dB.}$$

at $\omega = \omega_{c1}$; $A = 20\log\left[\frac{0.75}{j\omega}\right] = 20\log\left[\frac{0.75}{5}\right] = -16.45\text{ dB}$

at $\omega = \omega_{c2}$; ~~$A = 20\log[0.75$~~

$A = \left[(\text{change slope from } \omega_{c1} \text{ to } \omega_{c2}) \log\left[\frac{\omega_{c2}}{\omega_1}\right]\right] + A|_{\omega = \omega_{c1}}$

$A = 0 \times \log\left[\frac{\cancel{8}\,10}{5}\right] + (-16.5) = -16.5\text{ dB}$

at $\omega = \omega_h$; $A = \left[(\text{change slope from } \omega_{c2} \text{ to } \omega_h) \log\left[\frac{\omega_h}{\omega_{c2}}\right]\right] + A|_{\omega = \omega_{c2}}$

$A = \left[-40 \times \log\left[\frac{100}{10}\right]\right] + (-16.5) = -56.5\text{ dB}$

phasor plot:-

$$\angle G(j\omega) = \tan^{-1}(0.2\omega) - 90^\circ - \tan^{-1}\left[\frac{0.16\omega}{(1-0.01\omega^2)}\right]$$

$$= \tan^{-1}(0.2\omega) - 90^\circ - \tan^{-1}\left[\frac{0.16\omega}{(1-0.01\omega^2)} + \cancel{180^\circ}\right]$$

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}\left[\frac{0.16\omega}{(1-0.01\omega^2)} + \cancel{180}\right]$	$\angle G(j\omega)$
0.5	5.71	4.59	$-88.9 \approx -88$
1	11.31	9.18	$-87.9 \approx -88$
5	45	46.85	$-91.8 \approx -92$
10	63.44	90 ← $\tan^{-1}(\infty)$	$-116.6 \approx -116$
20	75.96	$-46.85 + 180 = 133.2$	$-147.3 \approx -148$
50	84.29	$-18.4 + 180 = 161.6$	$-167.3 \approx -168$
100	87.13	$-92 + 180 = 170.8$	$-173.7 \approx -174$

As the angle should be in increasing order we need to add 180°

phase margine $\gamma = 180 + \phi_{gc}$

$$= 180 - 88 = 92^\circ$$

gain margine $= \infty$

* Nyquist plot :-

1) Draw Nyquist plot for the system whose open loop transferfunction is

$$G(S)\,H(S) = \frac{K}{S(S+2)(S+10)} \quad (H(s)=1)$$

Sol

$$H(s)\,G(S) = \frac{K}{2\times10\times S(\frac{S}{2}+1)(\frac{S}{10}+1)} = \frac{0.05K}{S(0.5S+1)(0.1S+1)}$$

$$H(j\omega)G(j\omega) = \frac{0.05K}{j\omega(0.5j\omega+1)(0.1j\omega+1)}$$

$$|G(j\omega)H(j\omega)| = 0.05K/\left(\omega\sqrt{0.25\omega^2+1}\,\sqrt{0.01\omega^2+1}\right)$$

$$\angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(0.1\omega)$$

$$G(j\omega)H(j\omega) = \frac{0.05K}{j\omega(-0.05\omega^2+0.6j\omega+1)} = \frac{0.05K}{(-0.05j\omega^3-0.6\omega^2+j\omega)}$$

$$H(j\omega)G(j\omega) = \frac{0.05K}{-0.6\omega^2+j\omega(-0.05\omega^2+1)}$$

When the locus of $G(j\omega)H(j\omega)$ crosses real axis the imaginary term will be zero and corresponding freq is phase cross over frequency ω_{pc}

at $\omega = \omega_{pc}$ then $(1-0.05\omega^2)=0$

$\Rightarrow 1 = 0.05\omega_{pc}^2 \Rightarrow \omega_{pc} = 4.472$ rad/sec

Substitute $\omega_{pc} = 4.472$ rad/sec in the real part

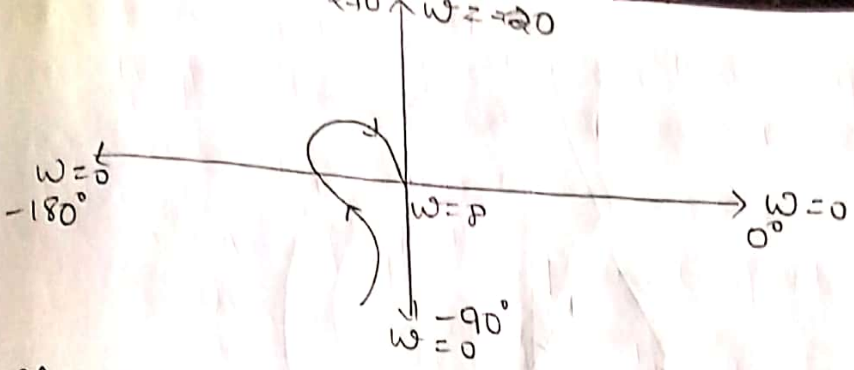
i.e $G(j\omega)H(j\omega) = \frac{0.05K}{-0.6\omega_{pc}^2} = \frac{0.05K}{-0.6\times4.472}$

$G(j\omega)H(j\omega) = +0.0047K.$

→ Section 1 :-

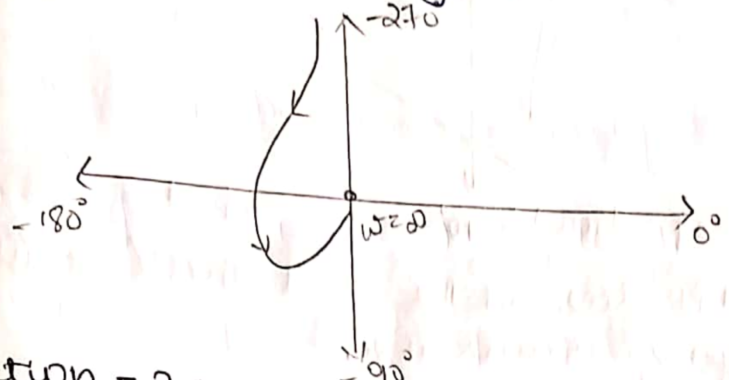
At $\omega=0$, $|G(j\omega)H(j\omega)| = \infty$, $\angle G(j\omega)H(j\omega) = -90^\circ$

At $\omega=\infty$, $|G(j\omega)H(j\omega)| = 0$, $\angle G(j\omega)H(j\omega) = -270^\circ$



→ section-3 :-

It is the mirror image of section-1.



→ section - 2 :-

$$G(S)\,H(S) = \frac{0.05K}{S(0.5S+1)(0.1S+1)} \qquad \because (1+ST) \approx ST$$

$$G(S)\,H(S) = \frac{0.05K}{S(0.5S)(0.1S)} = \frac{K}{S^3}$$

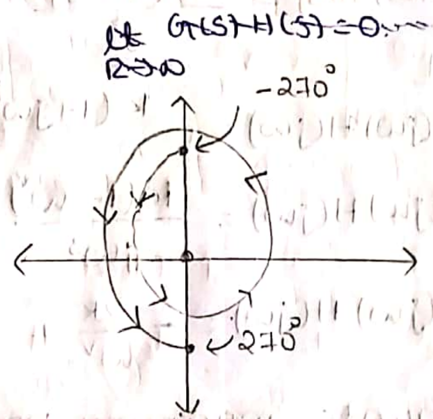
where $S = \lim_{R\to\infty} e^{j\theta}$ $G(S)\,H(S)$

at $\frac{\pi}{2}$, $0e^{-j3\theta} = 0e^{-j\frac{3\pi}{2}}$

$-\pi/2$, $0e^{-j3\theta} = 0e^{j\frac{3\pi}{2}}$

starts from $-270°$ and end at $+270°$

~~$\lim_{R\to\infty}$ G(S) H(S) = 0...~~



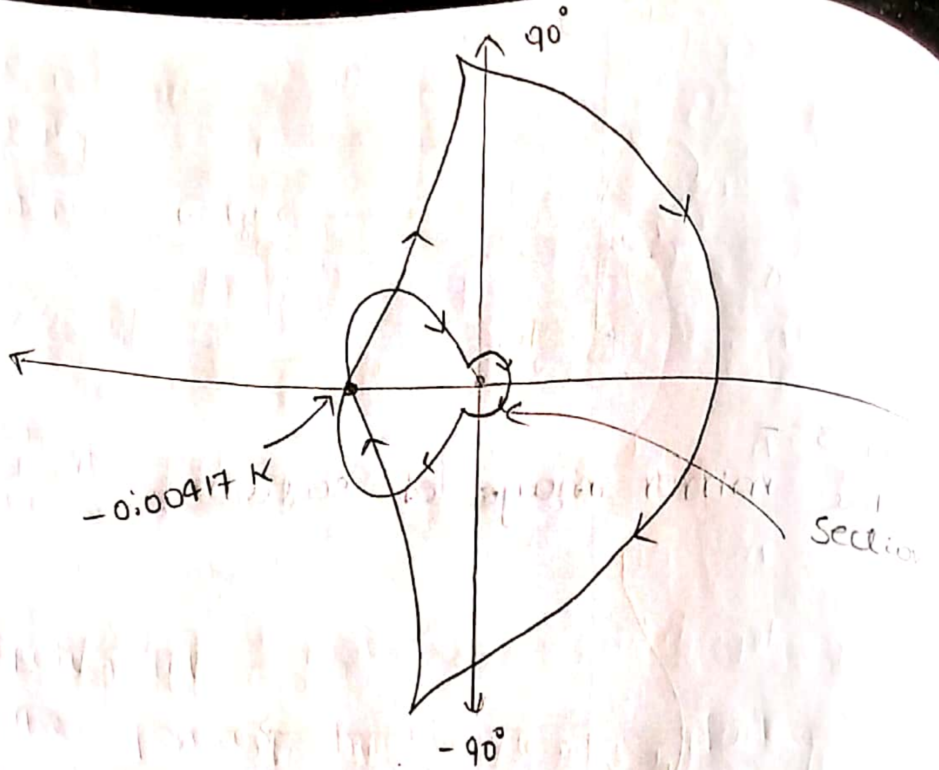
→ section - 4 :-

$\because (1+ST) \approx 1$ $\quad G(S)\,H(S) = \frac{0.05K}{S(1)(1)} = \frac{0.05K}{S}$

$S = \lim_{R\to 0} e^{j\theta}$.

at $-\frac{\pi}{2}$, $\infty e^{-j\theta} = \infty e^{+j\frac{\pi}{2}}$

at $\frac{\pi}{2}$, $\infty e^{-j\theta} = \infty e^{-j\frac{\pi}{2}}$



If $-0.00417K$ lies in the unit circle ie then the system is unstable.

(22/Mar/2019)

2) Construct Nyquist plot for a system whose loop transfer function is given by $G(s)H(s) = \frac{K(1+s)^2}{s^3}$. Find ranges of 'K'.

Sol

$$G(j\omega)H(j\omega) = K(1+j\omega)^2/-j\omega^3 \Rightarrow \cancel{K(1-\omega)^2}$$

$$G(j\omega)H(j\omega) = \frac{K(1-\omega^2)}{-j\omega^3} - \frac{2j\omega K}{j\omega^3}$$

$$G(j\omega)H(j\omega) = -\frac{2K}{\omega^2} + \frac{jK(1-\omega^2)}{\omega^3}$$

Equate imaginary part to zero

$K(1-\omega^2) = 0 \Rightarrow \omega = \pm 1$ rad/sec. $\therefore \omega_{pc} = \pm 1$ rad/s

Substitute ω_{pc} in real part

ie $G(j\omega)H(j\omega) = -\frac{2K}{(1)^2} = -2K$

→ ~~Section 1:-~~

$$|G(j\omega)H(j\omega)| = \frac{K\left(\sqrt{(1+\omega^2)}\right)^2}{\omega^3} = \frac{K(1+\omega^2)}{\omega^3}$$

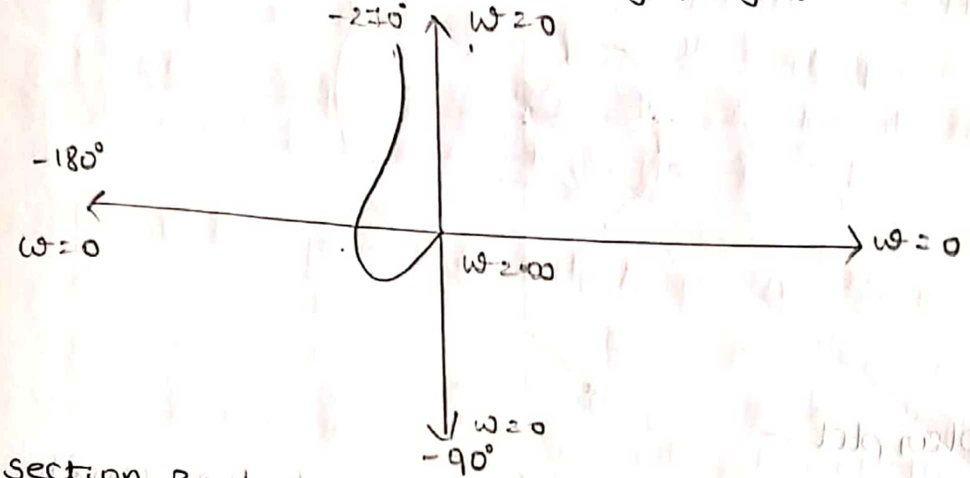
$$\angle G(j\omega)H(j\omega) = 2\tan^{-1}(\omega) - 270°$$

Note: $(1+jx)^n = A$ then $\angle A = n\tan^{-1}(x)$.

→ section 1 :-

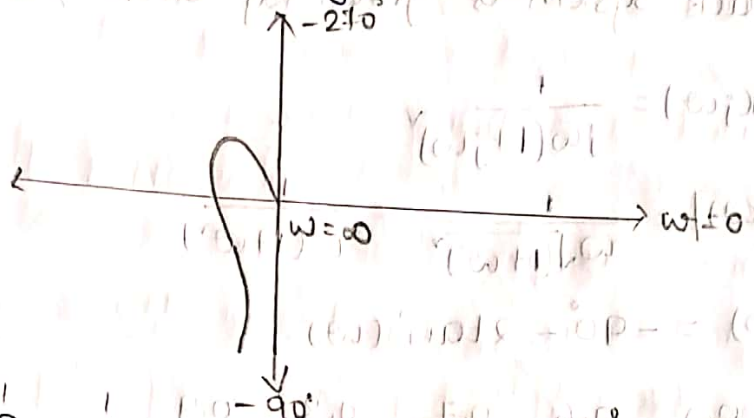
At $\omega=0$; $|G(j\omega)H(j\omega)| = \infty$, $\angle G(j\omega)H(j\omega) = -270^\circ$

At $\omega=\infty$; $|G(j\omega)H(j\omega)| = 0$, $\angle G(j\omega)H(j\omega) = -90^\circ$



→ section 3 :-

It is the mirror image of section 1



→ section 2 :-

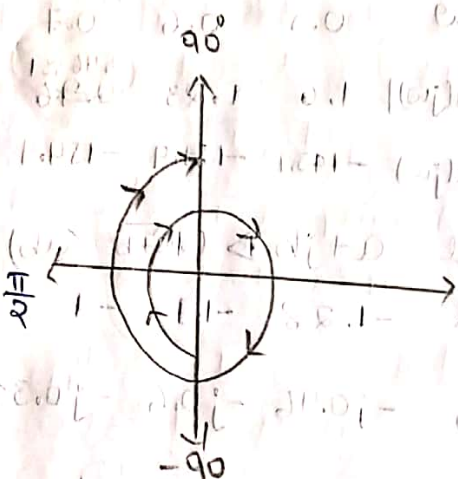
$(1+sT) \approx sT$

$G(s)H(s) = \frac{K(s)^2}{s^3} = \frac{K}{s}$

where $s = \lim_{R\to\infty} e^{j\theta}$.

at $\frac{\pi}{2}$; $0\,e^{-j\theta} = 0\,e^{-j\frac{\pi}{2}}$

at $-\frac{\pi}{2}$; $0\,e^{-j\theta} = 0\,e^{j\frac{\pi}{2}}$



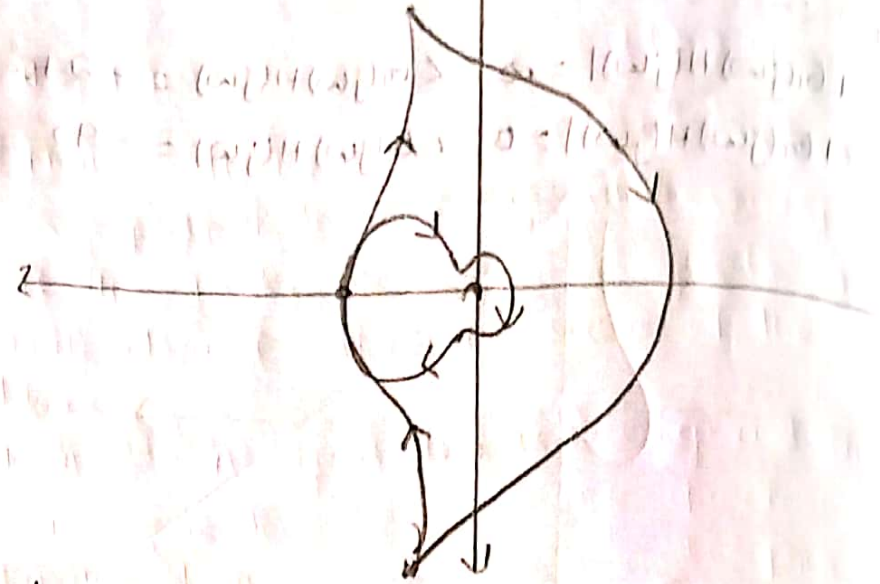
→ section 4 :-

$(1+sT) \approx 1$

$G(s)H(s) = \frac{K(1)^2}{s^3}$ where $s = \lim_{R\to 0} e^{j\theta}$

at $-\pi/2$; $\infty e^{-j3\theta} = \infty e^{j3\pi/2}$

at $\pi/2$; $\infty e^{-j3\theta} = \infty e^{-j3\pi/2}$



***polar plot :-**

1) The open loop transferfunction of the unity feedback system is given by $G(s) = 1/s(s+1)^2$

Sol

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)^2}} = \frac{1}{\omega(1+\omega^2)}$$

$$\angle G(j\omega) = -90^\circ - 2\tan^{-1}(\omega)$$

ω	0.5	0.6	0.7	0.8	0.9	1	1.1
$\lvert G(j\omega)\rvert$	1.6	1.23	0.96 ≈ 1 ~~0.76~~	0.76	0.61	0.5	0.411
$\angle G(j\omega)$	−143.1	−151.9	−159.9	−167.3	−173.9	−180	−185.95

let $a+jb \triangleright (\sqrt{a^2+b^2} \angle ab)$

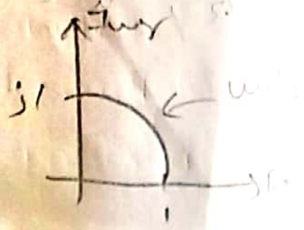
a	−1.28	−1.1	−1	−0.74	−0.6	−0.5	−0.41
jb	−j0.96	−j0.6	−j0.34	−j0.16	−j0.06	j0	+j0.0

$$\text{gain Margin} = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} \quad \because \omega_{pc} = -180^\circ$$

$$= \frac{1}{0.5} = 2\,dB$$

phase Margin $\gamma = 180 + \phi_{gc} = 180 - 160 = 20^\circ$

$\phi_{gc} = \angle G(j\omega)|_{\omega=\omega_{gc}}$ $\quad \because \omega_{gc} = 1$



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UNIT-V

FREQUENCY RESPONSE ANALYSIS

Frequency domain Specifications

The frequency domain Specifications are

1) Resonant peak (M_r)
2) Resonant frequency (ω_r)
3) Bandwidth (ω_b)
4) Cutoff rate
5) Gain margin (k_g)
6) phase margin (γ)

1) Resonant Peak (M_r):-

The maximum value of magnitude of closed loop transfer function is called Resonant peak (M_r).

2) Resonant frequency (ω_r):-

The frequency at which the resonant peak occurs is called resonant frequency (ω_r).

3) Bandwidth (ω_b):-

The frequency at which the gain is -3dB is Called Cut-off frequency. Bandwidth is usua--lly define for closed loop System and it transmits the signal whose frequencies are less than Cut-off frequency.

4) Cut-off rate:-

The slope of the magnitude curve nearly Cut-off frequency is Called Cut-off rate.

5) Gain margin (k_g):-

order to bring the System stable.

$\therefore$ Gain margin $(k_g) = \dfrac{1}{|G(j\omega_{pc})|}$

where ω_{pc} = phase Cross over frequency.

k_g in dB $= 20 \log \dfrac{1}{|G(j\omega_{pc})|}$

6) Phase margin(γ):-

the phase margin (γ) is defined as additional phase lag to be added at gain Cross over frequency in order to bring the system stable.

$\therefore$ phase margin $(\gamma) = 180^\circ + \phi_{gc}$

where ϕ_{gc} = the angle of $G(j\omega_{pc})$

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Frequency domain Specifications of Second order System:-

1) Resonant Peak (M_r):-

Consider the closed loop transfer function of second order System

$$\frac{C(S)}{R(S)} = M(S) = \frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

Replacing $S = j\omega$

$$\Rightarrow M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$\Rightarrow M(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\xi\omega\omega_n + \omega_n^2}$$

$$\Rightarrow M(j\omega) = \frac{\omega_n^2}{\omega_n^2\left[\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1\right]}$$

$$\Rightarrow M(j\omega) = \frac{1}{\quad}$$

Let normalised frequency $u = \frac{\omega}{\omega_n}$

$$\therefore M(j\omega) = \frac{1}{(1-u^2) + j2\xi_e u}$$

Let M = magnitude of closed loop transfer function

α = phase of closed loop transfer function

$$M = |M(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + 4\xi_e^2 u^2}} = \{(1-u^2)^2 + 4\xi_e^2 u^2\}^{-1/2} \quad —①$$

$$\alpha = \angle M(j\omega) = -\tan^{-1}\left(\frac{2\xi_e u}{(1-u^2)}\right)$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M w.r.t u and making $\frac{dM}{du} = 0$ when $u = u_r$

where $u_r = \frac{\omega_r}{\omega_n}$ = normalised resonant frequency.

Differentiating M w.r.t u

$$\therefore \frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\xi_e^2 u^2\right]^{-1/2}$$

$$\Rightarrow \frac{dM}{du} = -\frac{1}{2}\left[(1-u^2)^2 + 4\xi_e^2 u^2\right]^{-3/2} \cdot \left[2(1-u^2)(-2u) + 8\xi_e^2 u\right]$$

$$= \frac{-\left[-4u(1-u^2) + 8\xi_e^2 u\right]}{2\left[(1-u^2)^2 + 4\xi_e^2 u^2\right]^{3/2}}$$

when $u = u_r$, $\frac{dM}{du} = 0$

$$\frac{-\left[-4u_r(1-u_r^2) + 8\xi_e^2 u_r\right]}{2\left[(1-u_r^2)^2 + 4\xi_e^2 u_r^2\right]^{3/2}} = 0$$

$$\Rightarrow -4u_r[1-u_r^2] + 8\xi_e^2 u_r = 0$$

$$\Rightarrow -4[1-u_r^2] = -8\xi_e^2$$

$$\Rightarrow \boxed{u_r = \sqrt{1-2\xi_e^2}}$$

$\therefore$ The resonant peak occurs when $u_r = \sqrt{1-2\xi_e^2}$

when $u = u_r$; $M = M_r$.

$$\therefore M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + 4\xi_e^2 u_r^2}}$$

$$M_r = \frac{1}{\sqrt{\left[1-(1-2\xi_e^2)\right]^2 + 4\xi_e^2\left[1-2\xi_e^2\right]}}$$

$$M_r = \frac{1}{\left\{4\xi_e^4 + 4\xi_e^2(1-2\xi_e^2)\right\}^{1/2}}$$

$$M_r = \frac{1}{\left\{4\xi_e^4 + 4\xi_e^2 - 8\xi_e^4\right\}^{1/2}}$$

$$M_r = \frac{1}{\left[4\xi_e^2 - 4\xi_e^4\right]^{1/2}}$$

$$M_r = \frac{1}{\left[4\xi_e^2(1-\xi_e^2)\right]^{1/2}}$$

$$\therefore \boxed{M_r = \frac{1}{2\xi_e\sqrt{1-\xi_e^2}}}$$

2) Resonant frequency (ω_r):-

we know that normalised resonant frequency

$$u_r = \frac{\omega_r}{\omega_n} = \sqrt{1-2\xi_e^2}$$

$\therefore$ Resonant frequency $\boxed{\omega_r = \omega_n\sqrt{1-2\xi_e^2}}$

3) Bandwidth (ω_b):-

Let normalised bandwidth $u_b = \frac{\omega_b}{\omega_n}$

when $u = u_b$, $M = \frac{1}{\sqrt{2}}$

$$\therefore \frac{1}{\sqrt{2}} = \frac{1}{\left[(1-u_b^2)^2+4\xi_e^2u_b^2\right]^{1/2}}$$

$$\Rightarrow 2 = (1-u_b^2)^2+4\xi_e^2u_b^2$$

$$\Rightarrow 2 = 1+u_b^4-2u_b^2+4\xi_e^2u_b^2$$

$$\Rightarrow u_b^4+u_b^2\left[4\xi_e^2-2\right]-1=0$$

Let $x = u_b^2$

$$x^2+x\left[4\xi_e^2-2\right]-1=0$$

$$x = \frac{-(4\xi_e^2-2)\pm\sqrt{16\xi_e^4+4-16\xi_e^2+4}}{2}$$

$$x = \frac{-2(2\xi_e^2-1)\pm\sqrt{16\xi_e^4-16\xi_e^2+8}}{2}$$

$$x = \frac{-2(2\xi_e^2-1)\pm2\sqrt{4\xi_e^4-4\xi_e^2+2}}{2}$$

$$\therefore x = 1-2\xi_e^2\pm\sqrt{4\xi_e^4-4\xi_e^2+2}$$

Considering only positive sign.

$$x = 1-2\xi_e^2+\sqrt{4\xi_e^4-4\xi_e^2+2}$$

as $x = u_b^2 \Rightarrow u_b = \sqrt{x}$

$$\therefore u_b = \left[1-2\xi_e^2+\sqrt{4\xi_e^4-4\xi_e^2+2}\right]^{1/2}$$

As $u_b = \frac{\omega_b}{\omega_n} \Rightarrow \omega_b = \omega_n u_b$

$$\boxed{\omega_b = \omega_n\left[1-2\xi_e^2+\sqrt{4\xi_e^4-4\xi_e^2+2}\right]^{1/2}}$$

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4) Phase Margin (γ):-

The open loop transfer function of second order system is given by

Substitute $s = j\omega$

$$\therefore G(j\omega) = \frac{\omega_n^2}{j\omega[j\omega + 2\xi_e\omega_n]}$$

$$\Rightarrow G(j\omega) = \frac{\omega_n^2}{\omega_n^2\left[\frac{j\omega}{\omega_n}\cdot\frac{(j\omega+2\xi_e\omega_n)}{\omega_n}\right]}$$

$$\Rightarrow G(j\omega) = \frac{1}{j\left(\frac{\omega}{\omega_n}\right)\left[j\frac{\omega}{\omega_n}+2\xi_e\right]}$$

we know that

normalised frequency, $u = \frac{\omega}{\omega_n}$

$$\Rightarrow G(j\omega) = \frac{1}{ju[ju+2\xi_e]}$$

Magnitude of $G(j\omega)$

$$\text{i.e., } |G(j\omega)| = \frac{1}{\sqrt{u^2}\sqrt{4\xi_e^2+u^2}}$$

$$\Rightarrow |G(j\omega)| = \frac{1}{u\sqrt{4\xi_e^2+u^2}} \quad \text{— ①}$$

phase of $G(j\omega)$ is

$$\alpha = -\tan^{-1}\left(\frac{u}{0}\right) - \tan^{-1}\left(\frac{2\xi_e}{u}\right)\left(\frac{u}{2\xi_e}\right)$$

$$\alpha = -90^\circ - \tan^{-1}\left(\frac{2\xi_e}{u}\right)\left(\frac{u}{2\xi_e}\right) \quad \text{— ②}$$

At gain crossover frequency i.e., $u = u_{gc}$;

$$|G(j\omega)| = 1$$

$$\text{At } u = u_{gc} \Rightarrow |G(j\omega)| = \frac{1}{u_{gc}\sqrt{4\xi_e^2+u_{gc}^2}} = 1$$

$$u_{gc}\sqrt{4\xi_e^2+u_{gc}^2} = 1$$

$$u_{gc}^2\left[4\xi_e^2+u_{gc}^2\right] = 1$$

$$\Rightarrow x^2 + 4\xi_e^2 x - 1 = 0$$

$$\therefore x = \frac{-4\xi_e^2 \pm \sqrt{16\xi_e^4 + 4}}{2}$$

$$x = -2\xi_e^2 \pm \sqrt{4\xi_e^4 + 1}$$

Considering only positive sign

$$x = -2\xi_e^2 + \sqrt{4\xi_e^4 + 1}$$

As $x = u_{gc}^2 \Rightarrow u_{gc} = \sqrt{x}$

$$\Rightarrow u_{gc} = \left[-2\xi_e^2 + \sqrt{4\xi_e^4 + 1}\right]^{1/2}$$

as $u_{gc} = \frac{\omega_{gc}}{\omega_n} \Rightarrow \omega_{gc} = u_{gc}\omega_n$

$$\boxed{\omega_{gc} = \omega_n\left[-2\xi_e^2 + \sqrt{4\xi_e^4 + 1}\right]^{1/2}}$$

Phase margin is obtained when $u = u_{gc}$ and $\omega = \omega_{gc}$.

$\therefore$ Phase margin $\gamma = 180^\circ + \alpha \Big|_{u = u_{gc};\ \omega = \omega_{gc}}$

$$\Rightarrow \gamma = 180^\circ - 90^\circ - \tan^{-1}\left[\frac{\cancel{2\xi_e}}{\cancel{u_{gc}}}\right]\left[\frac{u_{gc}}{2\xi_e}\right]$$

$$\gamma = 90^\circ - \tan^{-1}\left[\frac{u_{gc}}{2\xi_e}\right]$$

$$\gamma = 90^\circ - \tan^{-1}\left[\frac{\left[-2\xi_e^2 + \sqrt{4\xi_e^4 + 1}\right]^{1/2}}{2\xi_e}\right]$$

Problems

1) Find resonant peak, resonant frequency and bandwidth of unity feedback system whose open loop transfer function is $G(S) = \frac{0.5}{S^2 + 3S + 2}$

Sol:- Given open loop transfer function

$$G(S) = \frac{0.5}{S^2 + 3S + 2}$$

∴ The closed loop transfer function

$$\frac{C(S)}{R(S)} = M(S) = \frac{G(S)}{1+G(S)H(S)} = \frac{G(S)}{1+G(S)}$$

$$\Rightarrow M(S) = \frac{\frac{0.5}{S^2+3S+2}}{1+\frac{0.5}{S^2+3S+2}}$$

$$\Rightarrow M(S) = \frac{0.5}{S^2+3S+2.5} \quad \text{— ①}$$

The generalised closed loop transfer function of second order system is given as

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi_e\omega_n S+\omega_n^2} \quad \text{— ②}$$

$$① \Rightarrow M(S) = \frac{5}{5} \times \frac{0.5}{S^2+3S+2.5}$$

$$\Rightarrow M(S) = \frac{1}{5}\left[\frac{2.5}{S^2+3S+2.5}\right] \quad \text{— ③}$$

By Comparing ② & ③, we get

$\omega_n^2 = 2.5$ $\qquad 2\xi_e\omega_n = 3$

$\omega_n = 1.58$ $\qquad \Rightarrow \xi_e = \frac{3}{2\omega_n}$

$$\Rightarrow \xi_e = \frac{3}{2 \times 1.58} = 0.94$$

∴ Resonant peak $M_r = \frac{1}{2\xi_e\sqrt{1-\xi_e^2}}$

$$M_r = \frac{1}{2 \times 0.94\sqrt{1-(0.94)^2}}$$

$$M_r = \frac{1}{2 \times 0.94 \times 0.34}$$

$$M_r = 1.56$$

Resonant Frequency $\omega_r = \omega_n\sqrt{1-2\xi_e^2}$

$$\omega_r = 1.58\sqrt{1-1.76}$$
$$\omega_r = 1.58\sqrt{-0.76}$$
$$\omega_r = j\,1.58 \times 0.87$$
$$\omega_r = j\,1.37 \text{ rad/sec}$$

Bandwidth $\omega_b = \omega_n\left[1-2\xi_e^2+\sqrt{4\xi_e^4+2-4\xi_e^2}\right]^{1/2}$

$$\omega_b = 1.58\left[1-2(0.94)^2+\sqrt{4(0.94)^4+2-4(0.94)^2}\right]^{1/2}$$
$$\omega_b = 1.58\left[-0.767+\sqrt{1.588}\right]^{1/2}$$
$$\omega_b = 1.58\left[-0.767+1.26\right]^{1/2}$$
$$\omega_b = 1.58\left[0.493\right]^{1/2}$$
$$\omega_b = 1.58 \times 0.702$$
$$\omega_b = 1.109 \text{ rad/sec}$$

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Frequency response plots

Frequency response analysis of Control systems Can be Carried either analytically or graphically. the Various graphical techniques available for frequency response analysis are

1) Bode plot
2) polar plot (nyquist plot)
3) Nichols plot
4) M and N Circles
5) Nichols chart

Bode Plot :-

1) Sketch Bode plot for the following transfer function and determine the System gain k for

Sol: Given the transfer function

$$G(s) = \frac{ks^2}{(1+0.2s)(1+0.02s)}$$

$$\omega_c = 5 \text{ rad/sec}$$

Let $k=1$ and Put $s=j\omega$

$$\Rightarrow G(j\omega) = \frac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Magnitude plot

The Corner frequencies are

$$\omega_{c_1} = \frac{1}{0.2} = 5 \text{ rad/sec.}$$

$$\omega_{c_2} = \frac{1}{0.02} = 50 \text{ rad/sec}$$

Term	Corner frequency rad/sec	slope db/dec	change in slope db/dec
$(j\omega)^2$		$+40$	
$\frac{1}{1+0.2j\omega}$	$\omega_{c_1} = 5$	-20	$40-20=20$
$\frac{1}{1+0.02j\omega}$	$\omega_{c_2} = 50$	-20	$20-20=0$

choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ and choose a high frequency ω_h $\omega_h > \omega_{c_2}$

Let $\omega_l = 0.5$ rad/sec

and $\omega_h = 100$ rad/sec

Let $A = |G(j\omega)|$ in db

Let us Calculate A at $\omega_l, \omega_{c_1}, \omega_{c_2}, \omega_h$

At $\omega = \omega_l$; $A = 20 \log|(j\omega)^2| = 20 \log|\omega^2|$

At $\omega = \omega_{c1}$; $A = 20\log|(j\omega)^2| = 20\log|\omega^2|$

$= 20\log|5^2| = 27.95 \approx 28\,db$

At $\omega = \omega_{c2}$; $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right]$

$+ A_{(at\ \omega = \omega_{c2})}$

$$A = \left[20 \times \log\frac{50}{5}\right] + 28 = 48\,db$$

At $\omega = \omega_h$; $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right]$

$+ A_{(at\ \omega = \omega_{c2})}$

$$A = \left[0 \times \log\frac{\omega_h}{\omega_{c2}}\right] + 48 = 48\,db$$

Phase Plot :-

The phase angle of $G(j\omega)$ is given by

$$\phi = \angle G(j\omega) = 2\tan^{-1}\left[\frac{\omega}{0}\right] - \tan^{-1}\left[\frac{0.2\omega}{1}\right] - \tan^{-1}\left[\frac{0.02\omega}{1}\right]$$

$$= 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$$

ω rad/sec	$\tan^{-1}(0.2\omega)$ deg	$\tan^{-1}(0.02\omega)$ deg	ϕ deg
0.5	5.7	0.5	$173.8 \approx 174^\circ$
1	11.3	1.14	167.56
5	45	5.71	129.29
10	63.4	11.3	105.3
50	84.2	45	50.8
100	87.13	63.4	29.47

Calculation of k

Given that gain crossover frequency

$\omega_c = 5$ rad/sec.

At gain crossover frequency, the gain at db should be zero, but the value of gain at $\omega = 5$

plot, gain in db of $-28db$ should be added.

The value of k is calculated by equating $20 \log k$ to $-28db$.

$$\therefore 20 \log k = -28$$

$$\log k = \frac{-28}{20}$$

$$k = 10^{-28/20} = 0.03.$$